

*Theorie der Pendelbewegung mit Rücksicht auf die Gestalt und Bewegung der Erde, von P. A. Hansen, Director der Sternwarte in Gotha. (Neueste Schriften der Naturforschenden Gesellschaft in Danzig, Fünften Bandes Erstes Heft. Danzig, 1853.)*

It is stated on the Title-page that this is a Prize Memoir, which was crowned by the Physical Society of Danzig on the 3d of January, 1853. It contains an investigation of the various circumstances which affect the motion of a pendulum at the surface of the earth, whether we suppose the pendulum to be a mathematical point or a physical agglomeration of particles. In a short introduction, M. Hansen has given an account of some of the more important results to which he was conducted by his researches on the subject.

The Memoir is divided into seven sections. In the first section the author investigates the equations of motion of the simple pendulum, taking into account the figure and motion of the earth. Besides its diurnal rotation, the displacement of the earth, whether arising from its annual motion round the sun, or from the motion of the whole solar system in space, is considered in this part of the inquiry. It is found, however, that in both of the last-mentioned cases, the effect produced upon the motion of the pendulum is insensible.

The investigation of the motion of the pendulum is reduced to that of a freely-moving body by representing the tension of the pendulum-rod as an indeterminate force acting in the direction of its length. Eliminating this force from the differential equations of motion of the pendulum, M. Hansen obtains the equations of motion of a body, whether falling freely or projected from a given point in space. Although there was no novelty in the step, he takes occasion to integrate the equations in these two cases, omitting the effects due to the resistance of the atmosphere and the curvature of the earth's surface. The results are somewhat curious. First, in the case of a body falling freely, it is shown that if the height be very great, the deviation towards the *south* will be considerable. For small heights the displacement of the body in that direction will be imperceptible, but on the other hand it will be found to deviate sensibly towards the *east*. Thus, if a body fall from a height of 100 mètres under the equator, the deviation towards the east will be  $0^m.02199$ , and for the same height in latitude  $+45^\circ$  the deviation towards the south will be only  $0^m.000001$ , a quantity which is quite insensible.\*

M. Hansen next investigates the motion of the body when it is projected in any azimuthal direction whatever, and at any inclina-

\* It is said that Oersted was the first who remarked that falling bodies, besides deviating towards the east, deviate also a little towards the south. The fact was confirmed by subsequent experiments, but it remained unexplained until the discussion occasioned by Foucault's experiment led to a knowledge of its true cause, viz. the diurnal rotation of the earth.—EDITOR.

tion with respect to the horizon ; and he assigns the formulæ which the expressions for the co-ordinates assume in the two extreme cases of the initial impulse being parallel and perpendicular to the horizon. When the impulse is wholly horizontal, he shows that to an observer considering the body from the origin of motion, it will deviate always from the *left* to the *right* of the plane of projection, no matter what be the direction of the latter in azimuth. When the direction of the impulse is wholly vertical, the body after an assignable time will return to a position in the horizontal plane, from which it was projected, situate a little towards the *west* of its original position. In this case, then, the effect is the reverse of that produced when the body is allowed to fall freely ; it is also considerably greater. Thus, if the body attains a maximum altitude of 100 mètres, the deviation towards the west under the equator will be  $0^m.264$ , whereas in the case of a body falling freely from the same height, the easterly displacement was found to be only  $0^m.022$ , or barely one-eleventh of the former quantity.

M. Hansen commences the second section by a process of integration which furnishes him with a complete differential equation of the first order, relative to the motion of the simple pendulum, and he then proceeds to investigate the differential equations of motion for the compound or physical pendulum. In this case the rotatory motion of the pendulous mass is referred to a system of rectangular co-ordinates having the origin at the point of suspension, one of these axes being conceived to pass through the centre of gravity of the pendulum. With a view to simplify the problem, M. Hansen supposes the axes of rotation to be *principal axes*, and he further assumes that the moments of inertia about the axes which are perpendicular to the line passing through the centre of gravity and the point of suspension, or third axis of rotation, are equal to each other. This limitation obviously includes among various forms of construction, the pendulum as usually employed in experiments relative to the rotation of the earth, which has generally consisted of a homogeneous spherical ball, suspended from a fixed point by a slender thread or wire.

By integrating the equations of motion, M. Hansen obtains two *complete* differential equations of the first order, and a third equation of a similar kind, the terms of which are *imperfectly* divested of the integral sign. From one of the former of these equations it appears that, independently of any initial rotatory movement, the pendulum in the course of its oscillations will acquire a motion of rotation about the axis passing through the centre of gravity and the point of suspension. It is shown further that this rotatory motion is not of an oscillatory nature, but has a tendency to go on continually in the same direction.

In the third section, M. Hansen obtains a first approximation to the complete integrals of the equations of motion of the pendulum. First, considering the times during which the successive oscillations are performed, he shows that they are modified not only by the diurnal rotation of the earth, but also *by the rotation*

of the pendulum about its own axis. The correction, however, which is due to each of these causes, is so small in all actual experiments with the pendulum as to be quite insensible. Supposing the pendulum to consist of a very small homogeneous spherical body, suspended by a thread or slender metallic wire, and to perform one of its oscillations in a second of time, the correction depending on the rotation of the earth would be less than the three thousand millionth part of a second. Even if the pendulum extended to the enormous length of 900 mètres, the correction to the time of oscillation—which would in this case amount to 30 seconds—would be less than the hundred thousandth part of a second. Similar results are obtained with respect to the correction depending on the rotation of the pendulum about its own axis, supposing the diameter of the pendulum relative to its length to be small, and the velocity of rotation to be inconsiderable. Thus, supposing the diameter of the pendulum to be one-fortieth of its length, as in the case of the original experiment of Foucault, M. Hansen found that for a correction amounting to the twenty thousandth part of a single oscillation, the pendulum would require to make forty complete revolutions upon its axis during the time of each oscillation; a velocity of rotation so considerable that there would be no difficulty in guarding against it.

M. Hansen proceeds next to consider the path described by the pendulum in space. Confining his investigation to terms of the second order, he finds that the plane of oscillation is subject to a continual variation in azimuth depending on the diurnal rotation of the earth, the direction of azimuthal motion being always from east to west through south, or contrary to the diurnal motion of the earth, and the velocity of revolution being equal to the angular velocity of the earth around its axis multiplied by the latitude of the place of observation.

It appears further, from the researches of M. Hansen, that a *rotatory movement of the pendulum about its own axis causes the plane of oscillation to vary continually in azimuth*. The effect produced in this case will tend to diminish or to increase the azimuthal variation depending on the diurnal rotation of the earth, according as the pendulum rotates upon its axis in the direction of the earth's diurnal rotation, or in the opposite direction.

The pendulum is constrained, by the circumstances above mentioned, to pursue a path which does not anywhere intersect the vertical line, passing through the point of suspension. M. Hansen describes its movement by stating it to revolve upon the surface of an elliptical cone, the axes of the base of which continually vary in position with respect to the meridian of the observer, and he assigns the conditions which determine the magnitude and position of the axes of the ellipse.

M. Hansen found, that by no means a very considerable rotation of the pendulum about its axis would be necessary to produce an azimuthal variation of the plane of oscillation equal to or even exceeding the effect depending on the diurnal rotation of the earth.

Hence it is manifest, that from this cause the plane of oscillation might revolve in a direction actually opposite to that in which it would otherwise revolve. The amount of azimuthal motion depends in this case, not on the absolute length of the pendulum, but on the proportion of the diameter to the length; and, *cæteris paribus*, it gradually becomes less as the diameter of the pendulum is diminished. M. Hansen accordingly recommends, that in all experiments of this kind, the pendulum should be constructed of some substance having a high specific gravity, such as lead; and he suggests that care should be especially taken to guard against a rotatory motion of the pendulum upon its axis.

M. Hansen next considers the case in which the pendulum revolves upon the surface of a cone with a circular basis. Supposing the rotation upon its axis to vanish, he finds that the time which the pendulum will take to accomplish a complete revolution upon the surface of the cone will be different, according as the pendulum revolves from east to west through south, or in the opposite direction. For any place of observation in the northern hemisphere the time of revolution will be less in the former case than it will be in the latter; for any place in the southern hemisphere the result will be the reverse of this.

The difference between the times of a complete revolution of the pendulum, according as it revolves in one or the other of the directions just mentioned, is found to be independent of the initial elongation of the pendulum from the vertical direction; it is also found to change very slowly when the basis of the cone differs only in a slight degree from a circle. Supposing the length of the pendulum to be 10 mètres and the latitude to be  $51^{\circ}$ , M. Hansen found that, in the case of a pendulum revolving upon the surface of a cone, the base of which is perfectly circular, the difference between the times of a complete revolution, according as it moves in one direction or in the other, amounts to  $0^{\text{s}}.00073$ . This is, indeed, a very small quantity. It may, however, be rendered perceptible by the accumulated effect of many revolutions, as the experiments of M. Bravais have shown.

M. Hansen finally deduces the following theorem from his first approximation to the integrals of the equations of motion of the pendulum:—“*If, at the beginning of the motion, a plane having a fixed position with respect to the mass of the pendulum be conceived to pass through the axis of the latter, and also through the vertical line [in which the point of suspension is situate], this plane will continue to pass through the vertical line at each successive maximum elongation of the pendulum, notwithstanding the azimuthal motion of the plane of oscillation; provided that, at the beginning of motion, no impulse communicating a rotatory motion about its axis be applied to the pendulous body.*” The plane referred to will, in fact, always pass through the vertical line, abstracting from consideration a small periodical deviation. “Thus,” says the author, “in like manner, as the moon—although from a different cause—always turns the same side towards the earth, with

the exception of the slight fluctuations due to the effect of libration, so does the pendulum exhibit the same surface to an eye situate in the vertical, and viewing it in any direction whatever."\*

In the fourth section M. Hansen investigates the effects deducible from further approximations to the integrals of the equations of motion. In this part of his researches he employs the theory of the variation of arbitrary constants.† From the second approximation he derives a term expressing an azimuthal motion of the plane of oscillation, and depending, for any given length of the pendulum, on the product of the maximum and minimum elongations from the vertical. If no lateral impulse be communicated to the pendulum at the commencement of the oscillations, the azimuthal motion of the plane of oscillation (or, which amounts to the same thing, the azimuthal motion of the axes of the ellipse), as already determined by the first approximation, will not be sensibly affected by the additional term. When this condition, however, is not satisfied, the result will be different, but it may be found by the aid of the following consideration. Since any lateral impulse directly affects the semi-minor axis of the ellipse, we are enabled, from observations of the magnitude of the latter in any experiment, to infer, *à posteriori*, the intensity of the lateral force, and also the quantity of the azimuthal motion of the plane of oscillation depending on its influence. Supposing the length of the pendulum to be  $10^m$ , and the greatest and least elongations from the vertical to be  $\frac{1}{9}$  and  $\frac{1}{400}$ ‡ respectively, M. Hansen finds that the additional quantity of azimuthal motion resulting from the second approximation would then amount, for any place of observation in Central Europe, to one-fourth of the effect depending on the rotation of the earth, as already determined by the first approximation. It is manifest, therefore, that a much smaller value of the minimum elongation than that above-mentioned would suffice to exercise a sensible influence on the motion of the plane of oscillation.§

The term above referred to is the only one among those fur-

\* From the terms in which M. Hansen alludes to this curious principle (*ich bekomme ferner einen neuen Satz*), he appears to think that it had been hitherto unknown. It is right to mention, however, that soon after the announcement of Foucault's experiment, the theorem in question was pointed out to be a necessary consequence of the rotation of the earth; and it was also said to have been fully verified by subsequent experiments with the pendulum. (See for example, *Phil. Mag.*, June 1851, also *Mec. Mag.*, vol. liv., p. 326.) It is not improbable, however, that M. Hansen may have been the first who demonstrated the theorem by an analytical process.

† In the *Philosophical Magazine* for October 1851, there is a paper by the Rev. J. A. Combe, "*On the Motion of the Apse Line in the Pendulum Oval*," containing an elegant application of the theory of the variation of arbitrary constants. The rotation of the earth, however, is not taken into consideration.

‡ These fractions are supposed to refer, not to the mètre, but to the length of the pendulum as the standard unit.

§ For an investigation of the effect produced on the motion of the axes of the ellipse (without reference to the rotation of the earth), when the oscillations are nearly circular, and when they are almost in the same plane, see two papers by Mr. Airy, in vols. xi. and xx. of the *Mem. R. Ast. Soc.*

nished by the second approximation which M. Hansen found to be capable of producing a sensible effect.

In the fifth section M. Hansen considers the effect of the resistance of the atmosphere on the motion of the pendulum. The investigation is conducted upon two distinct hypotheses respecting the law of resistance, namely, when it is proportional to the simple power of the velocity, and when it is proportional to the square of the velocity. The effect is also considered when the oscillations are nearly in the same plane, and when they are nearly circular. The results deduced from the two hypotheses are essentially different, insomuch that observations instituted for the purpose, might not improbably indicate the hypothesis which most nearly represents the actual effect.

The sixth section of M. Hansen's Memoir is devoted to an investigation of the effect of the torsion of the thread or wire by which the pendulum is suspended.

In the seventh section M. Hansen proposes a new form of the pendulum, designed to obviate certain disadvantages attending the usual mode of construction. He remarks, that if the pendulum be suspended in the ordinary way, as described by him in the preceding part of his Memoir, it is indispensable, in order to avoid other sources of disturbance than those which he has investigated, that the thread or wire by which the pendulum is suspended should be homogeneous in its structure, and that its transverse section should be everywhere exactly circular. Especial care should be taken that the thread, in consequence of the pressure at its upper end, should not lose its circular-cylindrical form, since, from this cause alone, the variation of the plane of oscillation would be different in different azimuths. The stiffness of the thread also exercises an influence upon the law of the diminution of the arcs of oscillation which would otherwise result from the resistance of the atmosphere. M. Hansen, accordingly, is of opinion, that the following construction of the pendulum, inasmuch as it is free from those disturbing influences, is preferable to the usual form.

Instead of a slender wire, let the pendulum be suspended by a metal rod, so thick as to preclude all possibility of torsion. Let the ball be attached to the lower end of the rod, so that, if the axis of the latter were prolonged, it would pass through the centre of gravity of the ball. Let the rod be connected at the upper end with an apparatus of metal, terminating underneath in a spherical segment of small radius, the centre of which also lies in the prolongation of the axis of the pendulum rod. This spherical segment, composed of hardened steel or hard stone, being placed upon a horizontal plane surface of the same material, will constitute the point of suspension, as in earlier apparatuses the pendulum was made to swing by being suspended upon a knife-edge resting on a horizontal plane. It is intended that, during the oscillations of the pendulum, the supporting spherical segment shall *roll*, but not *slide*. For small arcs—of which only there is here any question—this object will be attained by merely smoothing the surfaces of the

spherical segment and its supporting plane without communicating to them a high polish.

M. Hansen remarks, that the form of construction here proposed by him especially possesses the advantage that, when the oscillations lie in one plane, or nearly so, a rotation of the pendulum about its axis (which he had shown to arise readily from the azimuthal motion of the plane of oscillation depending on the diurnal rotation of the earth) is no longer possible. The remainder of the section is occupied with an investigation of the motion of such a pendulum.

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*On the Secular Variation of the Moon's Mean Motion.*

By J. C. Adams, Esq., M.A. F.R.S. &c.\*

The author remarks, that in treating a great problem of approximation, such as that presented to us by the investigation of the moon's motion, experience shows that nothing is more easy than to neglect, on account of their apparent insignificance, considerations which ultimately prove to be of the greatest importance. One instance of this occurs with reference to the secular acceleration of the moon's mean motion. Although this acceleration and the diminution of the eccentricity of the earth's orbit, on which it depends, had been made known by observation as separate facts, yet many of the first geometers altogether failed to trace any connexion between them, and it was not until he had made repeated attempts to explain the phenomenon by other means, that Laplace himself succeeded in referring it to its true cause.

The accurate determination of the amount of the acceleration is a matter of very great importance. The effect on the moon's place, of an error in any of the periodic inequalities, is always confined within certain limits, and takes place alternately in opposite directions within very moderate intervals of time, whereas the effect of an error in the acceleration goes on increasing for an almost indefinite period, so as to render it impossible to connect observations made at very distant times.

In the *Mécanique Céleste*, the approximation to the value of the acceleration is confined to the principal term, but in the theories of Damoiseau and Plana, the developements are carried to an immense extent, particularly in the latter, where the multiplier of the change in the square of the eccentricity of the earth's orbit, which occurs in the expression of the secular acceleration, is given to terms of the seventh order.

As these theories agree in principle, and only differ slightly in the numerical value which they assign to the acceleration, and as they passed under the examination of Laplace, with especial reference to this subject, it might be supposed that only some small numerical rectifications would be required in order to obtain a very exact determination of this value.

It has not been, therefore, without surprise, which he has no doubt will be shared by the Society, that the author has lately

\* Proc. of the Roy. Soc. June 16, 1853.